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Geometrically nonlinear free vibrations of simply supported isotropic thin circular plates

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Abstract

Nonlinear free axisymmetric vibration of simply supported isotropic circular plates is investigated by using the energy method and a multimode approach. In-plane deformation is included in the formulation. Lagrange's equations are used to derive the governing equation of motion. Using the harmonic balance method, the equation of motion is converted into a nonlinear algebraic form. The numerical iterative method of solution adopted here is the so-called linearized updated mode method, which permits the authors to obtain accurate results for vibration amplitudes up to three times the plate thickness. The percentage of participation of each out-of-plane basic function to the deflection shape and to the bending stress at the plate centre and of each in-plane basic function to the membrane stress at the centre are calculated in order to determine the minimum number of in- and out-of-plane basic functions to be used in order to achieve a good accuracy of the model. The nonlinear frequency, the nonlinear fundamental mode shape and their associated nonlinear bending and membrane stresses are determined at large amplitudes of vibration. The numerical results obtained here are presented and compared with available published results, based on various approaches and with the single-mode solution. The limit of validity of the single-mode approach is also investigated.

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Nomenclature

r, θ, z	cylindrical coordinates
U, W	in- and out-of-plane displacements of the middle plane point $(r, \theta, 0)$, respectively
$\varepsilon_r, \varepsilon_\theta$	radial and circumferential strains
V_b, V_m, V	bending, membrane and total strain energy, respectively
E	Young's modulus
ν	Poisson's ratio of the plate material
ρ	mass per unit volume of the plate material
a, h	radius and thickness of the circular plate, respectively
D	bending stiffness of the plate, $= Eh^3/12(1-\nu^2)$
T	kinetic energy
$w_i(r)$	ith out-of-plane basic function, $W(r,t) = w_i(r)q_i^w(t)$
$u_i(r)$	ith in-plane basic function, $U(r,t) = u_i(r)q_i^u(t)$
q_i^w	ith out-of-plane generalized coordinate: $q_i^w(t) = a_i \cos(\omega t)$
q_i^u	ith in plane generalized coordinate: $q_i^u(t) = b_i \cos^2(\omega t)$
p_i, p_o	number of in- and out-of-plane basic functions, respectively
$k_{ij}^1, m_{ij}^1, b_{ijkl}^1$	general terms of the rigidity tensor, the mass tensor and the fourth-order nonlinearity tensor, respectively, associated with the transverse displacement
k_{ij}^2, m_{ij}^2	general terms of the rigidity tensor and the mass tensor, respectively, associated with the in-plane displacement
c_{ijk}	general term of the third-order nonlinearity rigidity tensor representing the coupling between the in- and the out-of-plane displacements
b_{ijkl}^*	general term of the fourth-order non-dimensional nonlinearity rigidity tensor taking into account the influence of the in-plane displacement
d_{ijk}^*	general term of the third-order non-dimensional tensor allowing the calculation of the kth in-plane contribution coefficient b_k
r^*	non-dimensional radial coordinate, $r^* = r/a$
λ	thickness-to-radius ratio of the circular plate, $\lambda = h/a$
τ	non-dimensional time, $t = \tau(\rho h a^4/D)^{1/2}$
ω, ω^*	frequency and non-dimensional frequency parameter, respectively
β_i	the ith transverse eigenvalue parameter for a simply supported axisymmetric circular plate
$(\omega_\ell^*)_i$	the ith non-dimensional linear natural frequency of axisymmetric vibrations of simply supported circular plates: $(\omega_\ell^*)_i = \beta_i^2$
α_i	the ith in-plane eigenvalue parameter for a simply supported immovable axisymmetric circular plate
$\{A\}$	column matrix of out-of-plane contribution coefficients, $\{A\} = [a_1 a_2 \ldots a_{p_o}]^{\mathrm{t}}$
$[Knl^*]$	the non-dimensional nonlinear geometrical stiffness matrix
Cn, K	Jacobean elliptic function and complete elliptic integral of the first kind
ε	cubic nonlinearity parameter: $\varepsilon = \mu/(w_1^*(0))^2$, in which $\mu = 2b_{1111}^*/k_{11}^{1*}$
$w_{\max}^*$	maximum non-dimensional vibration amplitude
σ_{br}^*	surface radial bending stress
σ_{mr}^*	radial membrane stress
$*$	star exponent indicates non-dimensional parameters

1. Introduction

Thin plate structures are encountered in various modern engineering problems and they are often subjected to severe dynamic loading. This may result in large vibration amplitudes of these structures. It is well known that when the amplitude of vibration is of the same order of the thickness of the plate, a significant geometrical nonlinearity is induced. This induces an increase in the resonance frequencies and a change in the mode shapes with the amplitude of vibration. The most widely used nonlinear equations of motion for thin plates are the dynamic analogue of the von Kármán equations, which were derived by Herrmann in cartesian coordinates [1]. Due to the complexity of the governing coupled nonlinear partial differential equations involved, no exact solution is yet known. Hence, each problem has received a special treatment involving some particular approximations.

In most of the studies carried out on large vibration amplitudes of circular plates, the common approach has been to use an assumed space or time mode. In the assumed space mode method, a spatial function which satisfies the related boundary conditions is assumed and Galerkin's method is used to eliminate the space variable from the governing equations. The problem is then reduced, in the case of a one-term expansion (single-mode approach), to the well-known Duffing equation in time, which may be solved in terms of elliptic functions or using other methods, such as the harmonic balance method or the perturbation method. This technique has been used to obtain approximate solutions based on the von Kármán equations with the single-mode approach in Refs. [2–9]. In the assumed time function method, a simple harmonic function in time is assumed and is then eliminated from the equation of motion using the Kantorovich averaging procedure. The resulting nonlinear spatial boundary value problem is solved numerically. This technique has been used with von Kármán equations in Refs. [10–13]. Some other studies [14–18] were based on a simplified nonlinear equation, obtained by using Berger's hypothesis [19]. Also, different perturbation techniques were used in the study of nonlinear vibrations of circular plates [20–22]. Various problems of nonlinear vibrations of circular plates have been investigated by using finite element methods [23–30]. In conjunction with the theoretical investigations mentioned above, few experimental studies were conducted in order to better understand the nonlinear dynamic behaviour of circular plates [5,31,32].

In most of the continuum approaches cited above, the single-mode assumption permitted the obtaining of analytical solutions for the amplitude frequency dependence and the nonlinear forced frequency response function. However, this assumption has been shown both theoretically and experimentally to be inaccurate for clamped–clamped beams and fully clamped rectangular and circular plates in Refs. [33–37], since the mode shape thus assumed is amplitude independent and therefore leads to linear patterns of the bending stress rather than the nonlinear patterns. Multimode analyses are needed in order to determine accurately the amplitude-dependent nonlinear frequencies and the associated nonlinear mode shapes, especially for fully clamped boundaries. A multiple-mode approach (three modes) has been presented in Ref. [38], in which the steady-state axisymmetric free and forced response of clamped circular plates has been investigated by using the Galerkin procedure and the harmonic balance method. Very recently, the geometrically nonlinear free axisymmetric vibrations of clamped circular plates have been studied by using a multimode model, taking into account the coupling between the higher vibration modes [36,37]. The main feature of this multimodal model, which is based on

Hamilton's principle and spectral analysis, is that it makes the geometrically nonlinear effects appear, not only via the amplitude frequency dependence, which was the main purpose of most of the previous studies on nonlinear vibrations, but also via the dependence of the structure deflection shapes on the amplitude of vibration. This allows quantitative estimates of nonlinear stresses to be obtained in sensible regions of the structure, which may be of crucial importance in the fatigue life prediction of structures working in or exposed to a severe environment.

The present study focuses on the geometrically nonlinear free vibration of simply supported circular plates using a multimodal approach. The mathematical formulation has been established using Lagrange's equations and the harmonic balance method. Consequently, the large-amplitude vibration problem is reduced to a set of nonlinear algebraic equations in terms of the contribution coefficients of the out-of-plane basic functions only. This set represents a nonlinear eigenvalue problem, which reduces to the well-known linear eigenvalue problem derived from Rayleigh–Ritz analysis when the nonlinearity is omitted. The nonlinear eigenvalue problem needs to be solved iteratively. The nonlinear iterative procedure described in Refs. [39,40], known as the linearized updated mode method, is used here as a first approach for accurate determination of the nonlinear resonant frequencies, the deflection shapes and the distributions of the associated membrane and bending stresses, for the fundamental (axisymmetric) simply supported circular plate nonlinear mode shape, at various non-dimensional amplitudes. In an investigation of the suitability of the single-mode approach for the geometrically nonlinear vibrations of a simply supported immovable circular plate, the iterative solution is compared to the analytical solution based on the single-mode assumption. A detailed comparison concerning the amplitude frequency dependence, the bending and membrane stresses has been conducted in order to determine accurately the range of validity of the single-mode analytical solution.

2. General formulation

2.1. Mathematical model

Consider a circular plate of thin uniform thickness h and radius a that is simply supported along its edge. The cylindrical coordinate system is chosen such that the middle plane of the plate coincides with the $r\theta$-plane. The origin of the coordinate system is at the centre of the plate with the z-axis downward in the thickness direction, as shown in Fig. 1. The plate material is assumed to be elastic, homogeneous and isotropic.

In large-amplitude axisymmetric vibrations of circular plates, the non-vanishing components of the strain tensor are given by [10]

$$\varepsilon_r = \frac{\partial U}{\partial r} + \frac{1}{2}\left(\frac{\partial W}{\partial r}\right)^2 - z\frac{\partial^2 W}{\partial r^2}, \quad \varepsilon_\theta = \frac{U}{r} - \frac{z}{r}\frac{\partial W}{\partial r}, \tag{1}$$

where U is the middle plane in-plane radial displacement and W is the out-of-plane transverse displacement.

The total strain energy, V, of the circular plate is given as the sum of the strain energy due to bending (V_b) and the membrane strain energy induced by large deflections (V_m): $V = V_b + V_m$.

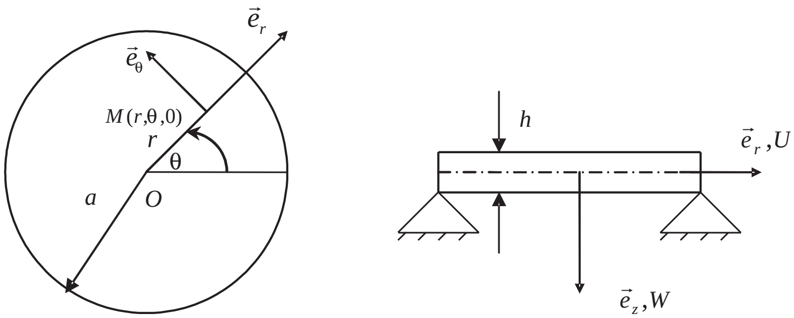

Fig. 1. Simply supported circular plate notation.

In the case of axisymmetric vibrations, the bending strain energy of the circular plate is given by [41]

$$V_b = \pi D \int_0^a \left[\left(\frac{\partial^2 W}{\partial r^2} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial r} \right)^2 + 2 \frac{v}{r} \frac{\partial W}{\partial r} \frac{\partial^2 W}{\partial r^2} \right] r \, \mathrm{d}r, \tag{2}$$

in which $D = Eh^3/12(1 - v^2)$ is the bending stiffness of the plate, and E and v are Young's modulus and Poisson's ratio of the plate material.

In terms of displacements, the expression for the membrane strain energy induced by large deflections for an axisymmetric circular plate is given by [41]

$$V_m = \frac{12\pi D}{h^2} \int_0^a \left[\left(\frac{\partial U}{\partial r} \right)^2 + \frac{U^2}{r^2} + 2v \frac{U}{r} \frac{\partial U}{\partial r} + \left(\frac{\partial W}{\partial r} \right)^2 \frac{\partial U}{\partial r} + \frac{1}{4} \left(\frac{\partial W}{\partial r} \right)^4 + v \frac{U}{r} \left(\frac{\partial W}{\partial r} \right)^2 \right] r \, \mathrm{d}r. \tag{3}$$

The total strain energy, V, is then given by

$$V = \pi D \int_0^a \left[\left(\frac{\partial^2 W}{\partial r^2} \right)^2 + \frac{1}{r^2} \left(\frac{\partial W}{\partial r} \right)^2 + 2 \frac{v}{r} \frac{\partial W}{\partial r} \frac{\partial^2 W}{\partial r^2} \right] r \, \mathrm{d}r + \frac{12\pi D}{h^2} \int_0^a \left[\left(\frac{\partial U}{\partial r} \right)^2 + \frac{U^2}{r^2} \right.$$
$$\left. + 2v \frac{U}{r} \frac{\partial U}{\partial r} + \left(\frac{\partial W}{\partial r} \right)^2 \frac{\partial U}{\partial r} + v \frac{U}{r} \left(\frac{\partial W}{\partial r} \right)^2 + \frac{1}{4} \left(\frac{\partial W}{\partial r} \right)^4 \right] r \, \mathrm{d}r. \tag{4}$$

The kinetic energy, T, of the circular plate is

$$T = \pi \rho h \int_0^a \left[\left(\frac{\partial W}{\partial t} \right)^2 + \left(\frac{\partial U}{\partial t} \right)^2 \right] r \, \mathrm{d}r \tag{5}$$

an expression in which rotatory inertia is neglected and ρ is the mass per unit volume of the plate material.

Using a generalized parameterization and the usual summation convention, one can put

$$W(r,t) = q_i^w(t) \, w_i(r), \quad i = 1, \ldots, p_o,$$
$$U(r,t) = q_i^u(t) \, u_i(r), \quad i = 1, \ldots, p_i, \tag{6}$$

where u_i and w_i are the in-plane and out-of-plane basic functions; q_i^u and q_i^w are the corresponding generalized parameters, respectively. p_i and p_o are the number of in- and out-of-plane basic functions used in the model. Insertion of Eq. (6) into Eqs. (4) and (5) leads to the following discretized expressions for the kinetic energy T and the total strain energy V:

$$T = \tfrac{1}{2}[\dot{q}_i^w \dot{q}_j^w m_{ij}^1 + \dot{q}_i^u \dot{q}_j^u m_{ij}^2], \tag{7}$$

$$V = \tfrac{1}{2}[q_i^w q_j^w k_{ij}^1 + q_i^w q_j^w q_k^w q_l^w b_{ijkl}^1 + q_i^w q_j^w q_i^u c_{ijk} + q_i^u q_j^u k_{ij}^2]. \tag{8}$$

In these equations, $m_{ij}^1, m_{ij}^2, k_{ij}^1, k_{ij}^2$ are the mass and rigidity tensors associated with W and U, respectively, b_{ijkl}^1 and c_{ijk} are, respectively, a fourth-order and a third-order nonlinearity tensors. The general terms of these tensors are given by

$$\begin{aligned}
m_{ij}^1 &= 2\pi\rho h \int_0^a w_i w_j r\,\mathrm{d}r, \quad m_{ij}^2 = 2\pi\rho h \int_0^a u_i u_j r\,\mathrm{d}r,\\
k_{ij}^1 &= 2\pi D \int_0^a \left(\frac{\mathrm{d}^2 w_i}{\mathrm{d}r^2}\frac{\mathrm{d}^2 w_j}{\mathrm{d}r^2} + \frac{1}{r^2}\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r} + \frac{\nu}{r}\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}^2 w_j}{\mathrm{d}r^2} + \frac{\nu}{r}\frac{\mathrm{d}^2 w_i}{\mathrm{d}r^2}\frac{\mathrm{d}w_j}{\mathrm{d}r} \right) r\,\mathrm{d}r,\\
k_{ij}^2 &= \frac{24\pi D}{h^2} \int_0^a \left(\frac{\mathrm{d}u_i}{\mathrm{d}r}\frac{\mathrm{d}u_j}{\mathrm{d}r} + \frac{1}{r^2} u_i u_j + \frac{\nu}{r}\frac{\mathrm{d}u_i}{\mathrm{d}r} u_j + \frac{\nu}{r} u_i \frac{\mathrm{d}u_j}{\mathrm{d}r} \right) r\,\mathrm{d}r,\\
c_{ijk} &= \frac{24\pi D}{h^2} \int_0^a \left(\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r}\frac{\mathrm{d}u_k}{\mathrm{d}r} + \frac{\nu}{r}\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r} u_k \right) r\,\mathrm{d}r,\\
b_{ijkl}^1 &= \frac{6\pi D}{h^2} \int_0^a \left(\frac{\mathrm{d}w_i}{\mathrm{d}r}\frac{\mathrm{d}w_j}{\mathrm{d}r}\frac{\mathrm{d}w_k}{\mathrm{d}r}\frac{\mathrm{d}w_l}{\mathrm{d}r} \right) r\,\mathrm{d}r.
\end{aligned} \tag{9}$$

It appears from Eq. (9) that the mass and rigidity tensors are symmetric, and the fourth-order tensor b_{ijkl}^1 and the third-order tensor c_{ijk} are such that

$$b_{ijkl}^1 = b_{klij}^1 = b_{jilk}^1 = b_{ikjl}^1, \quad c_{ijk} = c_{jik}. \tag{10}$$

By using Lagrange's equations and taking into account the properties of symmetry of the tensors involved, one can obtain the following set of coupled nonlinear differential equations:

$$\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(q_i^w) m_{ir}^1 + q_i^w k_{ir}^1 + 2 q_i^w q_j^w q_k^w b_{ijkr}^1 + q_i^w q_k^u c_{irk} = 0, \quad r = 1, \ldots, p_o,\\
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}(q_i^u) m_{is}^2 + q_i^u k_{is}^2 + \tfrac{1}{2} q_i^w q_j^w c_{ijs} = 0, \quad s = 1, \ldots, p_i.
\end{aligned} \tag{11}$$

Non-dimensional formulation is introduced by putting

$$w_i(r) = h w_i^*(r^*), \quad u_i(r) = \lambda h u_i^*(r^*), \quad t = \left(\frac{\rho h a^4}{D} \right)^{1/2} \tau, \tag{12}$$

where $r^* = r/a$ is the non-dimensional radial coordinate and $\lambda = h/a$ is a non-dimensional geometrical parameter representing the ratio of the plate thickness to its radius. Eq. (11) can be written in non-dimensional form as

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q_i^w)\, m_{ir}^{1*} + q_i^w\, k_{ir}^{1*} + 2q_i^w q_j^w q_k^w\, b_{ijkr}^{1*} + q_i^w q_k^u\, c_{irk}^* = 0, \quad r = 1, \ldots, p_o,$$
$$\lambda^2 \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q_i^u)\, m_{is}^{2*} + q_i^u\, k_{is}^{2*} + \tfrac{1}{2} q_i^w q_j^w\, c_{ijs}^* = 0, \quad s = 1, \ldots, p_i. \tag{13}$$

The m_{ij}^{1*}, m_{ij}^{2*}, k_{ij}^{1*}, k_{ij}^{2*}, c_{ijk}^* and b_{ijkl}^{1*} terms are non-dimensional tensors related to the dimensional ones by the following equations:

$$(m_{ij}^1,\, m_{ij}^2) = 2\pi\rho a^2 h^3\, (m_{ij}^{1*},\, \lambda^2\, m_{ij}^{2*}),$$
$$(k_{ij}^1,\, k_{ij}^2,\, c_{ijk},\, b_{ijkl}^1) = \frac{2\pi D h^2}{a^2}\, (k_{ij}^{1*},\, k_{ij}^{2*},\, c_{ijk}^*,\, b_{ijkl}^{1*}). \tag{14}$$

These non-dimensional tensors are defined by

$$m_{ij}^{1*} = \int_0^1 w_i^*\, w_j^*\, r^*\, \mathrm{d}r^*, \quad m_{ij}^{2*} = \int_0^1 u_i^*\, u_j^*\, r^*\, \mathrm{d}r^*,$$
$$k_{ij}^{1*} = \int_0^1 \left(\frac{\mathrm{d}^2 w_i^*}{\mathrm{d}r^{*2}} \frac{\mathrm{d}^2 w_j^*}{\mathrm{d}r^{*2}} + \frac{1}{r^{*2}} \frac{\mathrm{d}w_i^*}{\mathrm{d}r^*} \frac{\mathrm{d}w_j^*}{\mathrm{d}r^*} + \frac{v}{r^*} \frac{\mathrm{d}w_i^*}{\mathrm{d}r^*} \frac{\mathrm{d}^2 w_j^*}{\mathrm{d}r^{*2}} + \frac{v}{r^*} \frac{\mathrm{d}^2 w_i^*}{\mathrm{d}r^{*2}} \frac{\mathrm{d}w_j^*}{\mathrm{d}r^*} \right) r^*\, \mathrm{d}r^*,$$
$$k_{ij}^{2*} = 12 \int_0^1 \left(\frac{\mathrm{d}u_i^*}{\mathrm{d}r^*} \frac{\mathrm{d}u_j^*}{\mathrm{d}r^*} + \frac{1}{r^{*2}} u_i^* u_j^* + \frac{v}{r^*} \frac{\mathrm{d}u_i^*}{\mathrm{d}r^*} u_j^* + \frac{v}{r^*} u_i^* \frac{\mathrm{d}u_j^*}{\mathrm{d}r^*} \right) r^*\, \mathrm{d}r^*,$$
$$c_{ijk}^* = 12 \int_0^1 \left(\frac{\mathrm{d}w_i^*}{\mathrm{d}r^*} \frac{\mathrm{d}w_j^*}{\mathrm{d}r^*} \frac{\mathrm{d}u_k^*}{\mathrm{d}r^*} + \frac{v}{r^*} \frac{\mathrm{d}w_i^*}{\mathrm{d}r^*} \frac{\mathrm{d}w_j^*}{\mathrm{d}r^*}\, u_k^* \right) r^*\, \mathrm{d}r^*,$$
$$b_{ijkl}^{1*} = 3 \int_0^1 \frac{\mathrm{d}w_i^*}{\mathrm{d}r^*} \frac{\mathrm{d}w_j^*}{\mathrm{d}r^*} \frac{\mathrm{d}w_k^*}{\mathrm{d}r^*} \frac{\mathrm{d}w_l^*}{\mathrm{d}r^*}\, r^*\, \mathrm{d}r^*. \tag{15}$$

Upon neglecting the in-plane inertia, which is an acceptable assumption in most engineering applications of thin plates [42], the second set of equations in Eq. (13) can be solved for the in-plane generalized parameter q_i^u, leading to

$$q_k^u = q_i^w q_j^w\, d_{ijk}^*, \tag{16}$$

where $d_{ijk}^* = -\frac{1}{2} k_{kl}^{2*^{-1}} c_{ijl}^*$, is a new third-order tensor expressing the coupling between in-plane and transverse vibrations, in which $k_{ij}^{2*^{-1}}$ represents the inverse of the tensor k_{ij}^{2*}. Substituting Eq. (16) into the first set of Eq. (13) leads to a set of nonlinear differential equations in terms of the q_i^w's coefficients only

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q_i^w)\, m_{ir}^{1*} + q_i^w\, k_{ir}^{1*} + 2q_i^w q_j^w q_k^w\, b_{ijkr}^* = 0, \quad r = 1, \ldots, p_o. \tag{17}$$

Here, b_{ijkl}^* is a new fourth-order tensor given by

$$b_{ijkl}^* = b_{ijkl}^{1*} + \tfrac{1}{2} c_{ijn}^* d_{kln}^*, \tag{18}$$

which means that the term $(c_{ijn}^* d_{kln}^*)/2$ represents the contribution of the in-plane displacement to the nonlinearity.

2.2. Methods of solution

Eq. (17) represents a set of coupled Duffing's equations, for which an exact mathematical solution can be obtained only in the one-dimensional case corresponding to the single-mode approximation, in terms of elliptic functions.

2.2.1. Multimode solution

In the multidimensional case, an approximate solution may be obtained by the harmonic balance method. It is assumed here that harmonic motion exists for moderate finite vibration amplitudes. Such an assumption has been verified experimentally in Refs. [11,31]. Therefore, the out-of-plane generalized parameters can be written as

$$q_i^w(\tau) = a_i \cos(\omega^* \tau) = a_i \cos(\omega t), \tag{19}$$

where ω and ω^* are the dimensional and the non-dimensional nonlinear frequency parameters, respectively, which are related by

$$\omega^* = \left(\frac{\rho h a^4}{D}\right)^{1/2} \omega. \tag{20}$$

From Eq. (16), it appears that the in-plane generalized parameters can be written in the form

$$q_i^u(\tau) = b_i \cos^2(\omega^* \tau) = b_i \cos^2(\omega t), \tag{21}$$

where the in- and out-of-plane contribution coefficients are related by

$$b_k = a_i\, a_j\, d^*_{ijk}. \tag{22}$$

Substituting Eq. (19) into Eq. (17) and applying the harmonic balance method leads to

$$a_i\, k^{1*}_{ir} + \tfrac{3}{2} a_i a_j a_k\, b^*_{ijkr} - \omega^{*2} a_i\, m^{1*}_{ir} = 0, \quad r = 1, \ldots, p_o. \tag{23}$$

It is to be noted that Eq. (23) is identical to that obtained in Ref. [37] for the nonlinear free vibrations of clamped immovable circular plates by using Hamilton's principle and integration over the range $[0, 2\pi/\omega]$. The set of nonlinear algebraic equations (23) can be written in a matrix form as

$$([K^{1*}] + [Knl^*])\{A\} - \omega^{*2}[M^{1*}]\{A\} = \{0\}, \tag{24}$$

where $[M^{1*}]$, $[K^{1*}]$ and $[Knl^*]$ are the non-dimensional mass matrix, the non-dimensional linear stiffness matrix and the non-dimensional nonlinear geometrical stiffness matrix, respectively. Each term of the matrix $[Knl^*]$ is a quadratic function of the column matrix of coefficients $\{A\} = [a_1 a_2 \ldots a_{p_o}]^t$, and is given by: $(Knl^*)_{ij} = (3/2)\, a_k\, a_l\, b^*_{ijkl}$. It can be seen that when the nonlinear term is neglected, the nonlinear eigenvalue problem (24) reduces to the classical eigenvalue problem

$$[K^{1*}]\{A\} = \omega^{*2}[M^{1*}]\{A\}, \tag{25}$$

which is the Rayleigh–Ritz formulation of the linear vibration problem. In the linear case, the eigenvalue equation (25) leads to a series of eigenvalues and corresponding eigenvectors. In the nonlinear case, the solution of Eq. (24) should lead to a set of amplitude-dependent eigenvectors, with their amplitude-dependent associated eigenvalues. To solve the nonlinear eigenvalue problem (24), incremental-iterative methods are generally used. The iterative method of solution adopted here is that used in Refs. [39,43] for fully clamped isotropic and laminated rectangular plates, and in Ref. [40] for clamped–clamped beams. Very recently, this method of solution has been successfully used in Ref. [37] in order to determine the first two nonlinear mode shapes of clamped immovable circular plates. This method consists of solving successive linear eigenvalue problems by starting from the linear eigenvalue problem (25) until the convergence of the value of the eigenvalue ω^{*2} is achieved, leading also to the normalized eigenvector $\{A\}$, corresponding to the mode considered, according to the specified amplitude of vibration considered. It is to be noted that the nonlinear stiffness matrix $[Knl^*]$ is calculated in each iteration from the scaled eigenvector according to the specified amplitude of vibration obtained at the centre of the circular plate. For further details on this numerical iterative procedure, the reader is referred to Refs. [39,40].

For the r_0th (here $r_0 = 1$) nonlinear axisymmetric mode and for a given amplitude of vibration $w^*_{\max}$, the numerical iterative procedure determines accurately the non-dimensional nonlinear frequency parameter ω^* and the corresponding normalized eigenvector $\{A\}$, which in turn gives the r_0th nonlinear axisymmetric mode shape: $w^*(r^*) = a_i w^*_i(r^*)$, $i = 1, \ldots, p_o$. The corresponding in-plane shape function, i.e., $u^*(r^*) = b_i u^*_i(r^*)$, $i = 1, \ldots, p_i$, is determined by computing the in-plane contribution coefficients b_i from Eq. (22). Also, the associated nonlinear bending and membrane stresses can be determined quite easily.

2.2.2. Single-mode solution

The single-mode assumption consists of neglecting all of the coordinates except a single "resonant" coordinate. Thus, it reduces the multi-degree-of-freedom system to a single one. It has been shown in previous studies that such an assumption may not be very rigorous, with regard to some effects in nonlinear vibration of structures, such as the increase of curvatures near the clamps of a clamped–clamped beam [34], or the nonlinear increase of curvatures and bending stresses near to the edge of clamped circular plates [37]. However, the single-mode approach has been very often used in the literature [2–9]. This is due to the great simplification it introduces in the theory on one hand, and on the other hand because the error it introduces in the estimation of the nonlinear frequency remains very small for a large range of vibration amplitudes, as has been shown for example in Refs. [37,44]. The purpose here is to give explicit analytical expressions for the nonlinear frequency, the bending and membrane stresses. The two solutions, namely the iterative and the single-mode solutions will be compared in Section 3 in order to determine accurately the ranges of validity of the single-mode approach with respect to the nonlinear frequency, the bending and membrane stresses.

Applying the one-mode solution to Eq. (17) leads to

$$m^{1*}_{11} \frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q^w_1) + k^{1*}_{11} q^w_1 + 2b^*_{1111} q^{w3}_1 = 0. \tag{26}$$

This equation can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}(q^w_1) + \omega^{*2}_\ell (q^w_1 + \mu q^{w3}_1) = 0, \tag{27}$$

where $\omega^*_\ell = (k^{1*}_{11}/m^{1*}_{11})^{1/2}$ is the non-dimensional fundamental frequency and $\mu = 2b^*_{1111}/k^{1*}_{11}$.

Assuming that the amplitude of $q_1^w(\tau)$ is equal to a_1 at $\tau = 0$ and $\mathrm{d}q_1^w/\mathrm{d}\tau(0) = 0$, the exact mathematical solution of Eq. (27) can be given in terms of the Jacobean elliptic function Cn as [44]

$$q_1^w(\tau) = a_1\,\mathrm{Cn}(\gamma\,\tau,\,k), \tag{28}$$

where $\gamma^2 = \omega_\ell^{*2}(1 + \mu\,a_1^2)$ and $k^2 = \mu(\omega_\ell^{*2}a_1^2)/(2\gamma^2)$, in which k is the modulus of the elliptic function and γ may be taken as the "circular frequency". The elliptic function $\mathrm{Cn}(\gamma\,\tau,\,k)$ is periodic with a frequency $\omega^* = \pi\gamma/(2K(k))$, where $K(k)$ is the complete elliptic integral of the first kind. The amplitude frequency relation is then given by

$$\frac{\omega^*}{\omega_\ell^*} = \frac{\pi(1 + \mu\,a_1^2)^{1/2}}{2K(k)} \tag{29}$$

in which $k^2 = \mu a_1^2/(2 + 2\mu a_1^2)$.

Using a perturbation method, as in Ref. [44], for small values of a_1, the modulus k is also small and the elliptic function $\mathrm{Cn}(\gamma\,\tau,\,k)$ can be approximated by $\cos(\omega^*\,\tau)$. The first approximation of the exact solution is then given by

$$\left(\frac{\omega^*}{\omega_\ell^*}\right)^2 = 1 + \tfrac{3}{2}(b_{1111}^*/k_{11}^{1*})a_1^2. \tag{30}$$

It appears that the solution obtained from the first approximation of the elliptic function solution, i.e., Eq. (30), is identical to that obtained from Eq. (23), based on the harmonic balance method when specialized to the one-mode solution.

In terms of the non-dimensional amplitude of vibration $w_{\max}^* = a_1\,w_1^*(0)$, Eq. (30) can be rewritten as

$$\left(\frac{\omega^*}{\omega_\ell^*}\right)^2 = 1 + \tfrac{3}{4}\,\varepsilon\,(w_{\max}^*)^2, \tag{31}$$

where $\varepsilon = \mu/(w_1^*(0))^2$, is the cubic nonlinearity parameter [8].

The in-plane contribution coefficients are now given by

$$b_i = a_1^2\,d_{11i}^* \tag{32}$$

and the in- and out-of-plane shape functions are

$$u^*(r^*) = a_1^2\,d_{11i}^*\,u_i^*(r^*), \quad w^*(r^*) = a_1\,w_1^*(r^*) \tag{33}$$

from which the non-dimensional membrane and bending stresses can be calculated.

2.3. Numerical details for the simply supported immovable circular plate

The basic functions w_i^* to be used in the expansion series of w in Eq. (6) must satisfy all of the simply supported theoretical boundary conditions, i.e., zero displacement and zero radial flexural moment along the circular edge. Since the linear problem of free axisymmetric flexural vibrations of a simply supported circular plate has an analytical solution, the chosen basic functions w_i^* were taken as the linear free oscillation modes of the simply supported circular plate given by [45]

$$w_i^*(r^*) = A_i\left[J_0\left(\beta_i r^*\right) - \frac{J_0\left(\beta_i\right)}{I_0\left(\beta_i\right)} I_0\left(\beta_i r^*\right)\right], \tag{34}$$

where β_i is the ith real positive root of the transcendental equation

$$\frac{J_1(\beta)}{J_0(\beta)} + \frac{I_1(\beta)}{I_0(\beta)} = \frac{2\beta}{1-v}. \tag{35}$$

Here J_n and I_n are, respectively, the Bessel and the modified Bessel functions of the first kind and of order n. The parameter β_i is related to the ith non-dimensional linear frequency parameter $(\omega_\ell^*)_i$ of the plate by

$$\beta_i^2 = (\omega_\ell^*)_i. \tag{36}$$

Since Eq. (35) depends on the value of Poisson's ratio of the plate material, the numerical values of β_i are computed here numerically by solving Eq. (35) for a value of v equal to 0.3. The first six values are given in Table 1.

The basic functions u_i^* to be used in the expansion series of u in Eq. (6) must satisfy the immovable in-plane boundary condition, i.e. $u_i^*(r^* = 1) = 0$. Here, the expansion is made in terms of a convenient set of orthonormal functions by putting, as in [37]

$$u_i^*(r^*) = B_i J_1(\alpha_i r^*), \tag{37}$$

where α_i is the ith real positive root of the equation $J_1(\alpha) = 0$, from which the first six numerical values of α_i are computed and are listed in Table 1.

The in- and out-of-plane basic functions are normalized in such a manner that

$$m_{ij}^{1*} = \int_0^1 w_i^* \, w_j^* \, r^* \mathrm{d}r^* = \delta_{ij},$$
$$m_{ij}^{2*} = \int_0^1 u_i^* \, u_j^* \, r^* \mathrm{d}r^* = \delta_{ij}. \tag{38}$$

The basic functions w_i^* and u_i^* $(i = 1, \ldots, 6)$ are shown in Figs. 2 and 3, respectively. The parameters k_{ij}^{1*}, k_{ij}^{2*}, c_{ijk}^* and b_{ijkl}^{1*} involved in the model were computed numerically by using Simpson's rule with 160 steps in the range [0, 1].

Table 1
Numerical values of the simply supported immovable circular plate parameters α_i and β_i intervening in the ith in- and out-of-plane basic functions, respectively, for $i = 1, \ldots, 6$

i	α_i	β_i
1	3.83170597020751	2.22151953459224
2	7.01558666981562	5.45160570218317
3	10.17346813506272	8.61139102796216
4	13.32369193631422	11.76087250211648
5	16.47063005087763	14.90687907946484
6	19.61585851046824	18.05129413941753

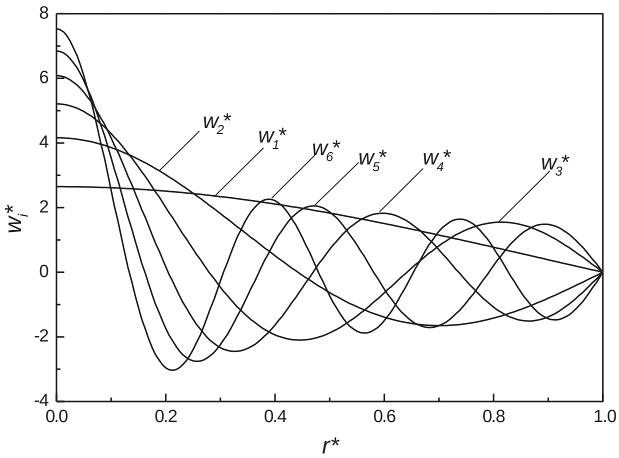

Fig. 2. Axisymmetric natural modes of vibration for a simply supported circular plate w_i^* for $i = 1, \ldots, 6$ ($v = 0.3$).

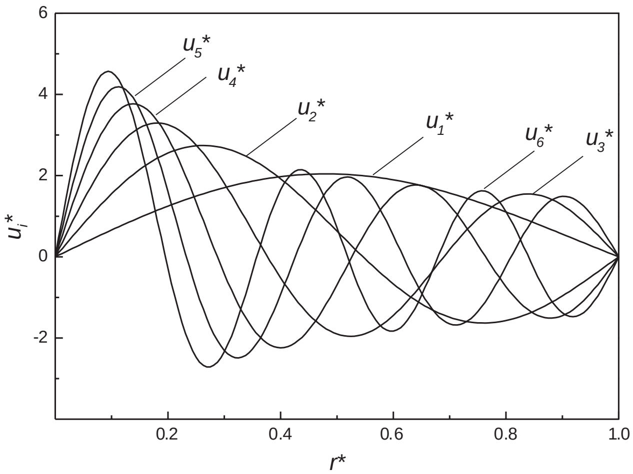

Fig. 3. The first six in-plane basic functions of a simply supported immovable circular plate.

3. Numerical results and discussion

3.1. Convergence study of the spectral expansion

The minimum number of in- and out-of-plane basic functions to be used in the present multimodal model, in order to achieve a good accuracy for large vibration amplitudes, have to be investigated first. It is to be noted here that the convergence criteria should not be restricted only to the nonlinear frequency, as was the case for example in Refs. [39,43,46], but must also involve the nonlinear bending and membrane stresses, in order to obtain reliable results with respect to engineering purposes. Concerning the fundamental nonlinear frequency, the participation of each linear mode to the maximum deflection can be defined by the following formula due to Shi et al. [47,48]:

$$\text{Participation from mode } i = 100\frac{|a_i|}{\sum_{r=1}^{p_o}|a_r|}. \tag{39}$$

In Table 2, the participation of each out-of-plane basic function to the first nonlinear mode shape of a simply supported immovable circular plate has been computed numerically for various maximum non-dimensional amplitudes, by using $p_i = p_o = 6$. For each maximum non-dimensional amplitude, obtained at the plate centre, the corresponding nonlinear frequency is given and compared with the single-mode solution. It can be seen that the single-mode approach yields accurate fundamental frequencies for maximum non-dimensional amplitudes up to 1.5. For much greater amplitudes of vibrations, at least a solution based on two linear transverse modes is required. The contributions of higher modes appear to be relatively small. However, their effects on the nonlinear bending stress estimates at the centre of the plate may not be negligible, since they intervene via their second derivatives, which are proportional to β_i^2. In order to examine this effect, another formula due to Azrar et al. [49] may be used to estimate the percentage of participation of each out-of-plane basic function to the non-dimensional maximum bending stress at the centre of the plate:

$$\text{Participation of } w_i^* \text{ to } \sigma_{\text{br}}^*(0) = 100\frac{|a_i|\beta_i^2}{\sum_{r=1}^{p_o}|a_r|\beta_r^2}. \tag{40}$$

The percentages of participation of each out-of-plane basic function to the non-dimensional maximum bending stress at the centre, $\sigma_{br}^*(0)$, for the fundamental nonlinear mode are given in Table 3. It can be seen that while the percentage of participation of the first out-of-plane basic function to the first nonlinear mode given in Table 2 remains predominant and greater than 99% for values of the non-dimensional amplitude $w_{\max}^*$ up to 1.5, its percentage of participation to the non-dimensional maximum bending stress at the centre is only 95.10% and decreases to about 90% for $w_{\max}^* = 3$. This shows that the influence of higher modes increases with the amplitude of vibration and that the consideration of the modal participation to the nonlinear

Table 2
Frequency ratio of free vibration and modal participation of a simply supported immovable circular plate and comparison with the single mode solution (42)

$W_{\max}^*$	ω_{nl}^*/ω_l^*	Modal participation (%)[a]						Single-mode solution (42)
		a_1	a_2	a_3	a_4	a_5	a_6	
0.2	1.0268	99.9793	0.0202	0.4114E−03	0.4234E−04	0.7703E−05	0.2219E−05	1.0268
0.5	1.1577	99.8762	0.1208	0.2683E−02	0.2714E−03	0.4990E−04	0.1373E−04	1.1572
1.0	1.5401	99.5702	0.4167	0.1167E−01	0.1159E−02	0.2127E−03	0.5726E−04	1.5351
1.5	2.0288	99.2072	0.7618	0.2751E−01	0.2784E−02	0.5115E−03	0.1355E−03	2.0130
2.0	2.5664	98.8746	1.0707	0.4830E−01	0.5148E−02	0.9561E−03	0.2511E−03	2.5350
2.5	3.1284	98.6020	1.3170	0.7098E−01	0.8092E−02	0.1534E−02	0.4026E−03	3.0787
3.0	3.7037	98.3881	1.5046	0.9308E−01	0.1138E−01	0.2218E−02	0.5855E−03	3.6344

[a]Participation of the ith out-of-plane basic function w_i^* to the deflection shape given by Eq. (39).

Table 3
Non-dimensional bending stress at the centre of a simply supported immovable circular plate and modal contributions at various amplitudes of vibration

W^*_{max}	$\sigma^*_{br}(0)$	Modal participation (%)[a]					
		$a_1\beta_1^2$	$a_2\beta_2^2$	$a_3\beta_3^2$	$a_4\beta_4^2$	$a_5\beta_5^2$	$a_6\beta_6^2$
0.2	0.3790	99.8703	0.1218	0.6175E−02	0.1185E−02	0.3465E−03	0.1464E−03
0.5	0.9410	99.2267	0.7225	0.4005E−01	0.7557E−02	0.2232E−02	0.9006E−03
1.0	1.8445	97.3306	2.4531	0.1715	0.3176E−01	0.9362E−02	0.3696E−02
1.5	2.7034	95.1004	4.3979	0.3962	0.7480E−01	0.2208E−01	0.8576E−02
2.0	3.5326	93.0564	6.0686	0.6831	0.1358	0.4052E−01	0.1560E−01
2.5	4.3484	91.3641	7.3488	0.9883	0.2101	0.6401E−01	0.2463E−01
3.0	5.1603	90.0123	8.2897	1.280	0.2917	0.9135E−01	0.3536E−01

[a]Participation of the ith out-of-plane basic function w_i^* to the bending stress at the plate centre given by Eq. (40).

mode, i.e. Eq. (39), may lead to inaccurate conclusions. From Table 3, it can be seen that at least four out-of-plane basic functions ($p_o = 4$) must be used for accurate determination of nonlinear bending stresses. In order to determine the minimum number of in-plane basic functions to be implemented in the model, a further formula is proposed here by considering the participation of each in-plane basic function to the membrane stress at the centre, as follows:

$$\text{Participation of } u_i^* \text{ to } \sigma^*_{mr}(0) = 100\,\frac{|b_i|\alpha_i}{\sum_{r=1}^{p_i}|b_r|\alpha_r}. \tag{41}$$

Formula (41) is based on the fact that each in-plane function u_i^* intervenes in the non-dimensional membrane stress at the plate centre via its first derivative, which is proportional to α_i. The percentages of participation of the in-plane functions to the non-dimensional membrane stress at the centre obtained using the new formula are summarized in Table 4. It can be seen that even for relatively small vibration amplitudes, the higher in-plane basic functions have a non-negligible contribution. For accurate determination of the membrane stress estimates for nondimensional vibration amplitudes up to three times the plate thickness, at least four in-plane basic functions are needed ($p_o = 4$).

In conclusion, accurate results for the geometrically nonlinear behaviour of simply supported immovable circular plates can be achieved by taking $p_i = p_o = 4$. For the single-mode approach, numerical results are obtained by letting $p_o = 1$ and $p_i = 4$.

3.2. Amplitude frequency dependence

The fundamental frequency ratios $\omega^*_{n\ell}/\omega^*_\ell$ at various maximum non-dimensional amplitudes w^*_{max} for a simply supported immovable circular plate are shown in Table 5. Comparison is made between results obtained by the multidimensional model, results obtained using the single-mode solutions, i.e., Eqs. (29) and (31), results obtained using an elliptic integral solution [2], and the finite element method [23,28]. It may be noted that the present results obtained by the multidimensional model agree very well with the elliptic solution [2] for maximum

Table 4
Non-dimensional membrane stress at the centre of a simply supported immovable circular plate and modal contributions at various amplitudes of vibration

W^*_{max}	$\sigma^*_{mr}(0)$	Modal participation (%)[a]					
		$b_1\alpha_1$	$b_2\alpha_2$	$b_3\alpha_3$	$b_4\alpha_4$	$b_5\alpha_5$	$b_6\alpha_6$
0.2	0.0359	98.6428	0.8513	0.1378	0.1552	0.1209	0.9204E−01
0.5	0.2248	99.1206	0.3663	0.1441	0.1556	0.1211	0.9220E−01
1.0	0.9014	98.3867	1.0621	0.1851	0.1559	0.1195	0.9076E−01
1.5	2.0345	96.6702	2.6769	0.2844	0.1617	0.1182	0.8858E−01
2.0	3.6276	95.1005	4.0827	0.4325	0.1778	0.1193	0.8711E−01
2.5	5.6819	93.8013	5.1839	0.6016	0.2032	0.1234	0.8660E−01
3.0	8.1975	92.7630	6.0159	0.7694	0.2347	0.1300	0.8700E−01

[a]Participation of the ith in-plane basic function u^*_i to the membrane stress at the plate centre given by Eq. (41).

non-dimensional amplitudes up to once the plate thickness, with a maximum difference of only 0.4%. The difference is more pronounced with numerical results obtained by the finite element method and linearizing procedure [28]. The present single-mode exact mathematical solution, given by Eq. (29), can be obtained by using the following numerical values of the simply supported immovable circular plate fundamental nonlinear mode shape modal parameters: $k^{1*}_{11} = 24.355696$ and $b^*_{1111} = 154.540891$. Since $w^*_1(0) = 2.648772$, the computed value of the nonlinear parameter ε appearing in Eq. (31) is 1.80877, which is slightly different from that obtained in Ref. [8], i.e., $\varepsilon = 1.85065$. This is due to the different choices of the spatial shape function. In Ref. [8], the authors used a transverse displacement function $w^*(r^*) = 1 - r^{*2}[(6+2v)+(1+v)r^{*2}]/(5+v)$. However, the present solution is thought to be more accurate since it is based on the exact linear mode shape. The present amplitude–frequency dependence based on the single-mode approach and the harmonic balance method is therefore given by

$$\frac{\omega^*}{\omega^*_\ell} = [1 + 1.35658\,(w^*_{max})^2]^{1/2}. \tag{42}$$

From Table 5, it is seen clearly that there are very small discrepancies between numerical results obtained by the multidimensional model and the single-mode solution (42) at large vibration amplitudes. The relative difference is only 1.22% at $w^*_{max} = 2.0$ and about 2% at $w^*_{max} = 3.0$. The influence of higher modes on the fundamental nonlinear frequency of a simply supported immovable circular plate can thus be neglected for maximum non-dimensional amplitudes up to twice the plate thickness.

3.3. Amplitude dependence of the fundamental nonlinear axisymmetric mode shape of simply supported immovable circular plates

Previous studies have shown that the mode shapes of beam-like and plate-like structures are amplitude dependent [10,34–37,39]. This effect is illustrated in the present case in Fig. 4, in which the normalized fundamental nonlinear mode shape of a simply supported immovable circular

Table 5
Frequency ratios of free vibration of a simply supported immovable circular plate

W^*_{max}	Present results						
	Iterative solution ($p_i = p_o = 4$)	Eq. (42) ($p_o = 1, p_i = 4$)	Elliptic integral (Eq. (29))	F.E.M. + Lin. [28]	F.E.M. + Lin. [23]	Elliptic integral [2,28]	Perturbation [8]
0.2	1.0268	1.0268	1.0267	1.0179	1.0263	1.0273	1.0274
0.4	1.1034	1.1032	1.1025	1.0700	1.1002	1.1047	1.1055
0.5	1.1577	1.1572	1.1557	—	—	—	—
0.6	1.2209	1.2200	1.2172	1.1518	1.2105	1.2217	1.2246
0.8	1.3693	1.3668	1.3606	1.2577	1.3455	1.3677	1.3741
1.0	1.5401	1.5351	1.5244	1.3826	1.4966	1.5342	1.5452
1.5	2.0288	2.0130	1.9887	—	—	—	—
2.0	2.5664	2.5350	2.4962	—	—	—	—
2.5	3.1285	3.0787	3.0257	—	—	—	—
3.0	3.7038	3.6344	3.5674	—	—	—	—

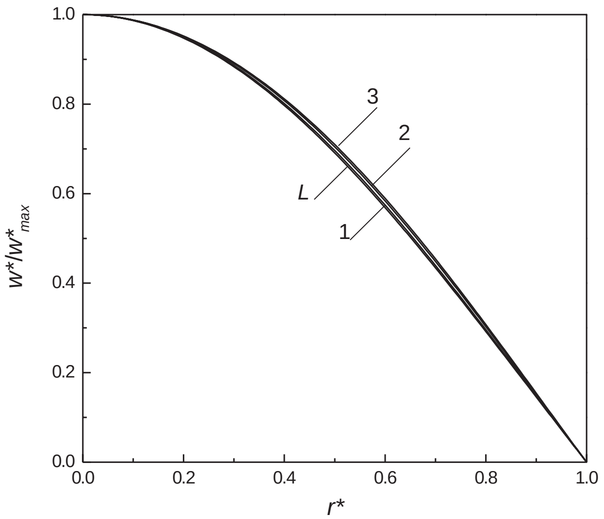

Fig. 4. Normalized radial sections of the fundamental linear and nonlinear mode shape of a simply supported immovable circular plate at various non-dimensional amplitudes. L, linear mode; 1, 2, 3, nonlinear mode shapes at amplitudes $w^*_{max} = 1.0$, 2.0 and 3.0, respectively.

plate is plotted for various values of the maximum non-dimensional amplitudes w^*_{max}. It can be seen clearly that the four curves are distinguishable, which means that the fundamental mode shape is amplitude dependent. Also, the general shape of the mode becomes flatter near to the centre of the circular plate with the increase in the amplitude of vibration. According to this fact, it can be expected that the nonlinear bending stress near to the plate centre will not increase as much as it does in the linear theory. This is due to the flattening of the shape which reduces the rate of increase of the curvature of the deformed shape compared to the linear mode shape. Such a situation has been encountered for a clamped–clamped beam [50], for fully clamped isotropic rectangular plates [39] and in the case of a clamped immovable circular plate [37].

3.4. Analysis of the radial bending and membrane stress distributions associated with the fundamental nonlinear axisymmetric mode shape

As shown previously, the present multimodal model enables one not only to determine the amplitude–frequency dependence, but also to determine the deformation of the mode shapes due to the geometrical nonlinearity. From the strength point of view, the accurate stress predictions of a structure undergoing large-amplitude vibration is sometimes more important than the prediction of resonant frequencies and mode shapes [39]. Fig. 5 shows the nondimensional surface radial bending stress distributions associated with the fundamental nonlinear mode shape of the simply supported circular plate, for various values of the vibration amplitude. It can be seen that the rate of increase of the bending stress near to the centre of the plate decreases as the amplitude increases. This fact is clearly shown in Fig. 6, in which the non-dimensional surface-bending stress at the centre is compared with the single-mode solution (equivalent also to the linear one). Also, at a maximum non-dimensional amplitude $w^*_{\max} = 1$, the difference between the linear and nonlinear maximum bending stress estimates obtained at the centre of the plate is only about 2.9% and increases to about 5.3% for $w^*_{\max} = 1.5$. The analytical solution of the maximum non-dimensional radial bending stress, obtained at the plate centre, from the single-mode approach is given by

$$\sigma^*_{\mathrm{br}}(0) = 1.897445\, w^*_{\max}. \tag{43}$$

Fig. 7 displays the non-dimensional radial membrane stress results, associated with the fundamental nonlinear mode shape at the centre and at the edge of the circular plate. In this

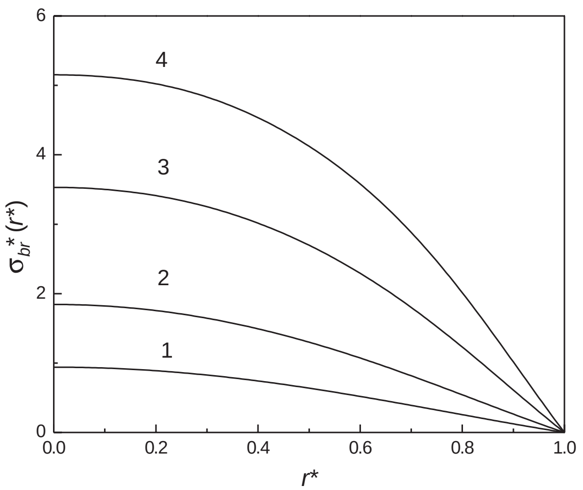

Fig. 5. Non-dimensional radial bending stress distribution associated with the simply supported immovable circular plate fundamental nonlinear mode shape at various non-dimensional amplitudes. 1, $w^*_{\max} = 0.5$; 2, $w^*_{\max} = 1.0$; 3, $w^*_{\max} = 2.0$; 4, $w^*_{\max} = 3.0$.

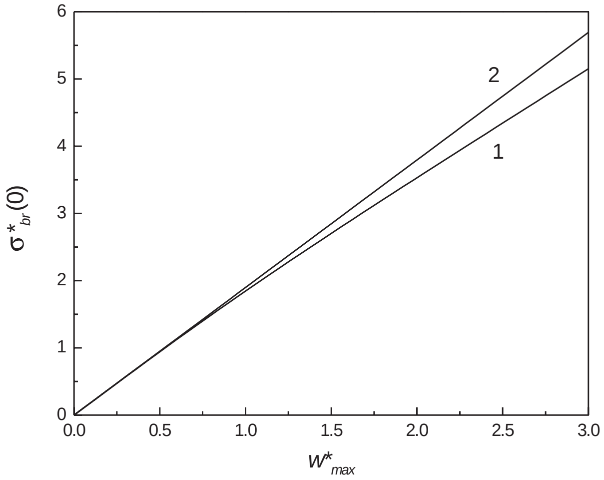

Fig. 6. Effect of large vibration amplitudes on the non-dimensional surface radial bending stress associated with the fundamental non-linear mode shape at the centre of a simply supported immovable circular plate. 1, Multimode solution; 2, linear (or single mode) solution.

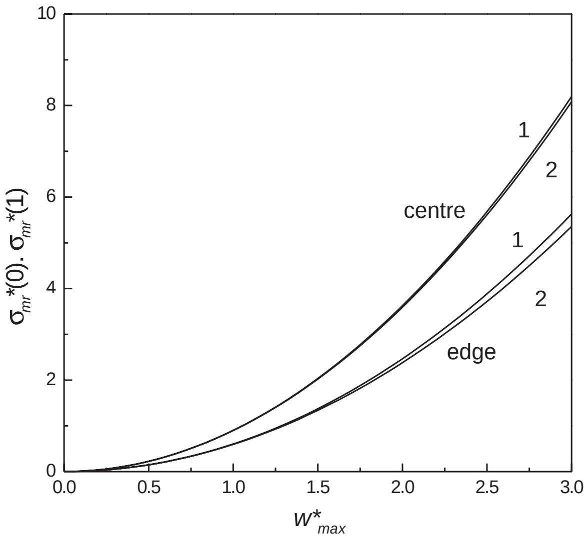

Fig. 7. Effect of large vibration amplitudes on the non-dimensional radial membrane stress associated with the fundamental nonlinear mode shape at the centre and at the edge of a simply supported immovable circular plate. 1, Multimode solution; 2, single-mode solution.

figure, the solutions obtained by the single-mode approach are also plotted. Examination of these curves shows a rapid increase in the membrane stress with increasing the amplitude of vibration, especially at the plate centre. It may be noted also that the single-mode approach gives very good estimates of the membrane stress. The analytical solution of the maximum non-dimensional radial membrane stress, obtained at the plate centre, from the single-mode approach is given by

$$\sigma^*_{mr}(0) = 0.897975\,(w^*_{max})^2. \quad (44)$$

The non-dimensional radial membrane stress distributions associated with the fundamental nonlinear mode shape are plotted in Fig. 8, for various values of the non-dimensional vibration amplitude. It can be seen that the membrane stress can be neglected at small vibration amplitudes. For example, the maximum non-dimensional radial membrane stress, obtained at the centre of the plate, for a maximum non-dimensional amplitude $w^*_{max} = 0.5$, is about 6.2% of the membrane stress, at the same location, for $w^*_{max} = 2.0$, and only 2.7% for $w^*_{max} = 3.0$. Furthermore, the maximum membrane stress for an amplitude of vibration equal to twice the plate thickness exceeds 50% of the maximum total stress and is about 61% for $w^*_{max} = 3.0$. This indicates that the membrane stress is very important in stress analysis and should not be neglected in engineering design of large deflected structures.

It is to be noted here that the present nonlinear results for the bending and membrane stresses obtained by the multimodal model and depicted in Figs. 6 and 7, corresponding to the simply supported immovable circular plates fundamental nonlinear mode shape, are in very good agreement with those obtained in Ref. [12] for non-dimensional amplitudes up to 2 (see Figs. 5 and 6 in this reference). In Ref. [12], von Kármán equations and the Kantorovich method were used, and numerical results were obtained by solving numerically a two-point boundary value problem. However, the present model, which leads to the numerical solution of a nonlinear eigenvalue problem, is quite interesting, due to its simplicity.

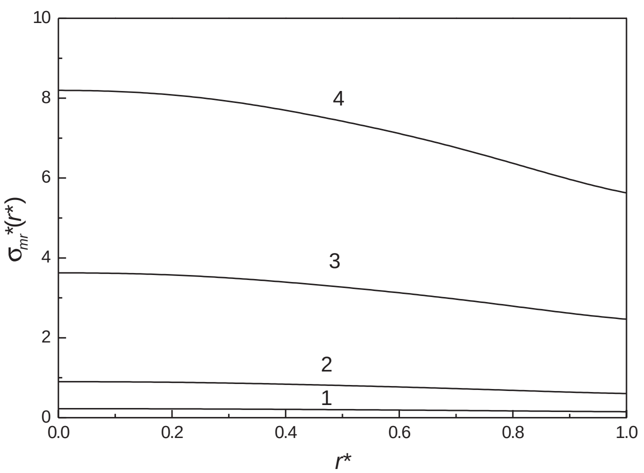

Fig. 8. Non-dimensional radial membrane stress distribution associated with the simply supported immovable circular plate fundamental non-linear mode shape for various non-dimensional amplitudes. 1, $w^*_{max} = 0.5$; 2, $w^*_{max} = 1.0$; 3, $w^*_{max} = 2.0$; 4, $w^*_{max} = 3.0$.

4. Conclusions

The geometrically nonlinear axisymmetric free vibration of a simply supported immovable thin isotropic circular plate has been examined theoretically in order to determine the effects of large vibration amplitudes on the fundamental axisymmetric mode shape and the corresponding natural nonlinear frequencies and associated membrane and bending stress distributions. The governing equations have been derived by using Lagrange's equations and the harmonic balance method. The in-plane deformation has been taken into account. When the in-plane inertia is neglected, the theory reduces the nonlinear free vibration problem to the solution of a set of nonlinear algebraic equations, in terms of the contribution coefficients of the transverse displacement only. This set of equations represents a nonlinear eigenvalue problem, which has been solved iteratively for each specified amplitude of vibration by the linearized updated mode method. The convergence study, based on determination of the modal participation to the deflection shape and to the non-dimensional bending and membrane stresses at the centre of the plate, has shown that accurate estimates of dynamic properties of geometrically nonlinear simply supported immovable circular plates can be achieved by using four in- and out-of-plane basic functions ($p_i = p_o = 4$).

Considering the results obtained, a hardening spring effect has been observed. Numerical results obtained here for the amplitude–frequency dependence are in good agreement with previous available results. It has also been shown that the geometrical nonlinearity induces a deformation of the fundamental nonlinear mode shape with the amplitude of vibration. In particular, as the amplitude of vibration increases, the mode shape becomes flatter near to the centre of the plate. As a direct consequence, the bending stresses near to the centre increase relatively slowly with the increase of the amplitude of vibration. The stress analysis has also shown a rapid increase of the membrane stress with increasing amplitude of vibration and a significant contribution of the membrane stress to the total stress at large vibration amplitudes. This indicates that the membrane stress is very important in stress analysis and should not be neglected in the engineering design of large deflected structures. Throughout the paper, the iterative (multimodal) solution has been compared to the single-mode approach solution. It has been found that the resonant nonlinear frequencies predicted by the single-mode approach are in good agreement with the iterative solution for a wide range of vibration amplitudes up to twice the plate thickness. It has also been shown that the single-mode approach gives accurate estimates of the maximum membrane stress. The only problem with the single-mode approach is that it is not able to predict the deformation of the mode shape of the simply supported immovable circular plate, which induces over-estimated maximum bending stresses. This limits the range of validity of this approach. As a conclusion, the single-mode approach is suitable for engineering design of simply supported immovable circular plates undergoing vibrations of finite amplitudes up to once their thickness.

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